Higher Schur-multiplicator of a Finite Abelian Group

by

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Abstract

In this paper we obtain an explicit formula for the *higher Schur-multiplicator* of an arbitrary finite abelian group with respect to the variety of nilpotent groups of class at most $c \geq 1$.

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1. Introduction and Preliminaries

In 1907 I.Schur [7] proved that the Schur-multiplicator of a direct product of two finite groups is isomorphic to the direct sum of the Schur-multiplicators of the direct factors and the tensor product of the two groups abelianized. (see also J.Wiegold [8].) If

$$G = \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \ldots \mathbf{Z}_{n_k}$$

is a finite abelian group , where $n_{i+1}|n_i$, for all $1 \le i \le k-1$, then using the above fact one obtains the Schur-multiplicator of G as follows (see [4]):

$$M(G) \cong \mathbf{Z}_{n_2} \oplus \mathbf{Z}_{n_3}^{(2)} \oplus \ldots \oplus \mathbf{Z}_{n_k}^{(k-1)} \quad ,$$

where $\mathbf{Z}_n^{(m)}$ denotes the direct product of m copies of the cyclic group $\mathbf{Z}_n$.

Now, in this paper, a similar result will be presented for the *higher Schur-multiplicator* of an arbitrary finite abelian group with respect to the variety of nilpotent groups of class at most $c \ge 1$, $\mathcal{N}_c$, say. (see [5] for the notation.)

Let G be any group with a free presentation

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1 \quad ,$$

where F is a free group. Then, following Heinz Hopf [3],

$$\frac{R \cap F'}{[R, F]} \quad ,$$

is isomorphic to the Schur-multiplicator of G , denoted by $M(G)$. Now , the *higher Schur-multiplicator* of G with respect to the variety of nilpotent groups of class at most $c \geq 1$, is defined to be

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R,\ _cF]} \quad ,$$

where $\gamma_{c+1}(F)$ is the $(c+1)$st-term of the lower central series and $[R,\ _cF]$ denotes $[R, \underbrace{F, F, \ldots, F}_{c-times}]$, see [5 or 6] for further details.

Let $H_i =< x_i | x_i^{r_i} > \cong \mathbf{Z}_{r_i}$, $i = 1, 2, \ldots, t$, $r_i \in \mathbf{N}$ be cyclic groups of order r_i , $1 \leq i \leq t$, $r_i \geq 0$ and let

$$1 \longrightarrow R_i =< x_i^{r_i} > \xrightarrow{\pi_i} F_i =< x_i > \longrightarrow H_i \longrightarrow 1$$

be the free presentation for H_i , where $1 \leq i \leq t$. Also, let

$$G = \prod_{i=1}^{t}{}^{\times} H_i = \mathbf{Z}_{r_1} \times \mathbf{Z}_{r_2} \times \ldots \times \mathbf{Z}_{r_t}$$

be the direct product of cyclic groups $\mathbf{Z}_{r_i}$'s. Then

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

is the free presentation for G, where

$$F = \prod_{i=1}^{t}{}^{*} F_i =< x_1, \ldots, x_t > \quad and \quad R =< x_i^{r_i}, \gamma_2(F) \cap [F_i]^* | i = 1, \ldots, t >^F \ ,$$

where $\prod_{i=1}^{t}{}^{*} F_i$ is the free product of F_i's , $i = 1, 2, \ldots, t$, and $[F_i]^*$ is the normal closure of some commutator subgroups of the the free product , defined

as follows:

$$[F_i]^* =< [F_i, F_j] \mid 1 \leq i, j \leq t \ , \ i \neq j \ >^F \quad .$$

Since F_i's are cyclic, we have $\gamma_2(F) \subseteq [F_i]^*$. Hence

$$R =< x_i^{r_i}, \gamma_2(F) \mid i = 1, \ldots, t >^F \ .$$

Now, put $S =< x_1^{r_1}, \ldots, x_t^{r_t} >^F$ and for all $m \geq 1$, define $\rho_1(S) = S$, $\rho_m(S) = [S, \ _{m-1}F]$, inductively. This yeilds the central series

$$S = \rho_1(S) \supseteq \rho_2(S) \supseteq \rho_3(S) \supseteq \ldots \supseteq \rho_m(S) \supseteq \ldots \quad .$$

Thus we have

$$R = S\gamma_2(F) \quad \text{and} \quad [R, \ _mF] = \rho_{m+1}(S)\gamma_{m+2}(F) \quad . (*)$$

Let $F = \prod_{i=1}^{t}{}^{*} F_i$ be the free product of $F_1, F_2, \ldots, F_t$. We define a *basic commutator subgroup* , $B(F_1, F_2, \ldots, F_t)_s$ of weight s ($s \in \mathbf{N}$) on t free groups $F_1, F_2, \ldots, F_t$, as follows:
We first order the subgroups $F_1, F_2, \ldots, F_t$ by setting $F_i < F_j$ if $i < j$. Then $B(F_1, F_2, \ldots, F_t)_s$ is the subgroup generated by all the basic commutators of weight s on t letters $x_1, x_2, \ldots, x_t$, where $x_{\in} F_i$ for all $1 \leq i \leq t$. For the definition of basic commutators see M.Hall [1].

Note that here we have slightly modified the definition of basic commutator subgroups from M.R.R.Moghaddam [6].

Now, let $T(H_1, H_2, \ldots, H_t)_s$ denote the summation of all the tensor products corresponding to the basic commutator subgroups $B(F_1, F_2, \ldots, F_t)_s$

where

$$1 \longrightarrow R_i \longrightarrow F_i \xrightarrow{\pi_i} H_i \longrightarrow 1$$

is the free presentation for H_i , $i = 1, 2, \ldots, t$. More precisely, if $[F_j.F_i, \ldots]$, with any bracketing, is a basic commutator subgroup of weight s in F_i's, then the *"corresponding"* tensor product will be

$$(H_j \otimes H_i \otimes \ldots) \quad ,$$

bracketed in the same way . (Note that H_i's are abelian groups.)

Similarly, the element $[x_j, x_i, \ldots]$ of the commutator subgroup $B(F_1, F_2, \ldots, F_t)_s$, with any bracketing , corresponds to the element of the tensor product

$$(\pi_j x_j \otimes \pi_i x_i \otimes \ldots)$$

bracketed in the same way, where x_k is the generator of F_k .

We keep this notation throughout the rest of the paper , and it will be used without further reference.

2. The Main Results

Lemma 2.1

Let G be a finite abelian group , then by the previous notation for all $c \geq 1$,

$$\mathcal{N}_c M(G) \cong \frac{\gamma_{c+1}(F)}{\rho_{c+1}(S)\gamma_{c+2}(F)} \quad .$$

Proof.

Let $1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$ be a free presentation for G , then by the definition and using the fact that $\gamma_{c+1}(F)$ is contained in $S\gamma_2(F) = R$, and $(*)$,

$$\begin{aligned} \mathcal{N}_c M(G) &\cong \frac{R \cap \gamma_{c+1}(F)}{[R,\ _cF]} \\ &\cong \frac{\gamma_{c+1}(F)}{[R,\ _cF]} \\ &\cong \frac{\gamma_{c+1}(F)}{\rho_{c+1}(S)\gamma_{c+2}(F)} \quad . \ \square \end{aligned}$$

Now, we are in a position to prove the following important theorem.

Theorem 2.2

Consider the above assumption and notation . Then , for all $c \geq 1$,

$$\frac{\gamma_{c+1}(F)}{\rho_{c+1}(S)\gamma_{c+2}(F)} \cong T(H_1, H_2, \ldots, H_t)_{c+1} \quad .$$

Proof.

Since F_i's are infinite cyclic groups , using Hall's Theorem [2] we obtain

$$B(F_1, F_2, \ldots, F_t)_{c+1} = \gamma_{c+1}(F) \qquad \textit{modulo } \gamma_{c+2}(F) \quad .$$

By the notation of previous section , there exists an obvious relation between $B(F_1, F_2, \ldots, F_t)_{c+1}$ and $T(H_1, H_2, \ldots, H_t)_{c+1}$. Also the commutators of weight $c+1$ in $B(F_1, F_2, \ldots, F_t)_{c+1}$ *modulo* $\gamma_{c+2}(F)$ are multilinear. Now

one may construct a homomorphism μ from $\gamma_{c+1}(F)/\rho_{c+1}(S)\gamma_{c+2}(F)$ into the abelian group $T(H_1, H_2, \ldots, H_t)_{c+1}$ given by

$$\prod \underbrace{[f, g, \ldots]}_{wt.c+1} \rho_{c+1}(S)\gamma_{c+2}(F) \longmapsto \sum(\pi_j f, \pi_i g, \ldots)$$

with any bracketing "*corresponding*" *element* .

We know that $\gamma_{c+1}(F)/\gamma_{c+2}(F)$ is the free abelian group on the basic commutators of weight $c+1$ on t letters , (by P.Hall [2]) . Now we define the same mapping, as the above correspondence, on the set of basic commutators of weight $c+1$ on t letters into the abelian group $T(H_i, H_2, \ldots, H_t)_{c+1}$. Then by the universal property of free abelian groups it can be extended to a homomorphism ϕ, say, from $\gamma_{c+1}(F)/\gamma_{c+2}(F)$ into $T(H_1, H_2, \ldots, H_t)_{c+1}$.

To show that ϕ induces the homomorphism μ from $\gamma_{c+1}(F)/\rho_{c+1}(S)\gamma_{c+2}(F)$ into $T(H_1, H_2, \ldots, H_t)_{c+1}$, we must prove that $\rho_{c+1}(S)$ is mapped onto zero under ϕ . But this is obvious, since

$$\underbrace{[x_i^{r_i}, f, g, \ldots]}_{wt.c+1} \equiv \underbrace{[x_i, f, g, \ldots]^{r_i}}_{wt.c+1} \quad (mod\ \gamma_{c+2}(F))$$

and

$$[x_i, f, g, \ldots]^{r_i} \stackrel{\phi}{\longmapsto} r_i(\pi_i x \otimes \pi_j f \otimes \pi_k g \otimes \ldots)$$

$$= (r_i \pi_i x \otimes \pi_j f \otimes \pi_k g \otimes \ldots) = 0 \quad .$$

Hence μ is the required homomorphism, which is also onto.

Conversely, by using the universal property of tensor product, we can define λ to be the homomorphism from $T(H_1, H_2, \ldots, H_t)_{c+1}$ into $\gamma_{c+1}(F)/\rho_{c+1}(S)\gamma_{c+2}(F)$ given by

$$\sum \underbrace{(h \otimes k \otimes \ldots)}_{(c+1)\ times} \longmapsto \prod \underbrace{[f, g, \ldots]}_{wt.c+1} \rho_{c+1}(S)\gamma_{c+2}(F) \quad ,$$

with any bracketing *bracketed the same way*

where for $h \in H_i$, $k \in H_j$, ..., we pick $f \in F_i, g \in F_j, \ldots$, such that $\pi_i f = h$, $\pi_j g = k$, ... Clearly, this is a well-defined map since the commutators on the right hand side are multilinear. One can easily see that λ is an epimorphism.

Now the result follows , since $\mu\lambda$ and $\lambda\mu$ are the identity maps on $T(H_1, H_2, \ldots, H_t)_{c+1}$ and $\gamma_{c+1}(F)/\rho_{c+1}(S)\gamma_{c+2}(F)$, respectively. $\square$

In some aspect , the following theorem is a generalization of I.Schur [7,4], J.Wiegold [8] , and M.R.R.Moghaddam [6] , where its proof follows from Lemma 2.1 and Theorem 2.2 .

Theorem 2.3

Let $\prod_{i=1}^{t}{}^{\times} H_i$ be the direct product of finite cyclic groups . Then by the above notation , the higher Schur-multiplicator of G is as follows :

$$\mathcal{N}_c M(\prod_{i=1}^{t}{}^{\times} H_i) \cong T(H_1, H_2, \ldots, H_t)_{c+1} \quad .$$

Now we are ready to give an explicit formula for the higher Schur-multiplicator of a finite abelian group with respect to the variety of nilpotent groups of

class at most $c \geq 1$, $\mathcal{N}_c$.

Let G be an arbitrary finite abelian group , then by the fundamental theorem of finitely generated abelian groups , $G \cong \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \ldots \oplus \mathbf{Z}_{n_k}$, where $n_{i+1}|n_i$ for all $1 \leq i \leq k-1$ and $k \geq 2$. $\mathbf{Z}_n^{(m)}$ will denote the direct product of m copies of the cyclic group $\mathbf{Z}_n$. Then with the above assumption, we obtain the following theorem , which is a vast generalization of I.Schur (see [4 or 7]).

Theorem 2.4

Let $G = \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \ldots \oplus \mathbf{Z}_{n_k}$ be a finite abelian group . Then , for all $c \geq 1$, the higher Schur-multiplicator of G is

$$\mathcal{N}_c M(G) \cong \mathbf{Z}_{n_2}^{(b_2)} \oplus \mathbf{Z}_{n_3}^{(b_3-b_2)} \oplus \ldots \oplus \mathbf{Z}_{n_k}^{(b_k-b_{k-1})} \quad ,$$

where b_i is the number of basic commutators of weight $c+1$ on i letters.

Proof.

Clearly $\mathbf{Z}_m \otimes \mathbf{Z}_n \cong \mathbf{Z}_{(m,n)}$ for all $m, n \in \mathbf{N}$, where (m, n) is the greatest common divisor of m and n . Hence

$$\mathbf{Z}_{m_1} \otimes \mathbf{Z}_{m_2} \otimes \ldots \otimes \mathbf{Z}_{m_k} \cong \mathbf{Z}_{m_k} \ ,$$

when $m_{i+1}|m_i$ for all $1 \leq i \leq k-1$. Thus

$$T(\mathbf{Z}_{n_1}, \mathbf{Z}_{n_2})_{c+1} \cong \mathbf{Z}_{n_2} \quad .$$

Now, by induction hypothesis assume

$$T(\mathbf{Z}_{n_1}, \mathbf{Z}_{n_2}, \ldots, \mathbf{Z}_{n_{k-1}})_{c+1} \cong \mathbf{Z}_{n_2}^{(b_2)} \oplus \mathbf{Z}_{n_3}^{(b_3-b_2)} \oplus \ldots \oplus \mathbf{Z}_{n_{k-1}}^{(b_{k-1}-b_{k-2})} \quad .$$

Then we have

$$T(\mathbf{Z}_{n_1}, \mathbf{Z}_{n_2}, \ldots, \mathbf{Z}_{n_k})_{c+1} = T(\mathbf{Z}_{n_1}, \mathbf{Z}_{n_2}, \ldots, \mathbf{Z}_{n_{k-1}})_{c+1} \oplus L \ ,$$

where L is the summation of all those tensor products of $\mathbf{Z}_{n_1}, \mathbf{Z}_{n_2}, \ldots, \mathbf{Z}_{n_k}$ corresponding to the basic commutators of weight $c+1$ on k letters which involve $\mathbf{Z}_{n_k}$. Since $n_k | n_i$ for all $1 \le i \le k-1$, all those tensor products are isomorphic to $\mathbf{Z}_{n_k}$. So L is the direct product of $(b_k - b_{k-1})$ copies of $\mathbf{Z}_{n_k}$. Hence the result follows, by induction. $\Box$

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